Notes on measurement sharpness

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Abstract

Some notes on measurement sharpness and its resource theory.

1 Introduction

While quantum theory has been traditionally developed around the concept of observables as self-adjoint operators and their spectral decompositions into orthogonal projections [1], it is a well-known fact that many fundamental problems, such as optimal joint measurements of noncommuting observables, and applications, such as optimal parameter estimation, require a more general formalism, where orthogonal projective decompositions of the identity are replaced by positive operator-valued measures, i.e., POVMs [2, 3, 4].

If POVMs constitute a notion of "approximate" observables, it is a natural question to ask, given a POVM, how close that is to an observable. This question has led several researchers to consider the problem of formalizing a concept of "sharpness" as a way to provide a quantitative measure of how close a given POVM is to a proper observable [5, 6, 7, 8, 9], where the latter is of course taken as the prototype of a perfectly sharp measurement.

In the light of recent developments in quantum information theory, it seems natural [10] to characterize the concept of sharpness within the framework of quantum resource theories [11], but attempts to construct a resource theory of sharpness have so far been unsuccessful. In this work we fill this gap by proposing a complete and operational resource theory of sharpness. The picture that we obtain is that greatest elements, i.e., sharp POVMs,

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exist, and that they coincide with POVMs that admit a repeatable measurement. Among these, conventional non-degenerate observables are singled out as the "minimal" ones. This is perfectly in line with what one would expect from a resource theory of sharpness. However, our sharpness resource theory possesses some extra desirable features providing connections with several areas of independent interest. First of all, the sharpness measures (in jargon, the *monotones*) that we introduce are defined using Ozawa's degrees of measurements correlation [12, 13] and, as such, are all in principle experimentally accessible. Second, it is shown that the class of sharpness-non-increasing transformations (in jargon, the class of *free operations*) corresponds to a restricted class of preprocessing channels applied not to the given POVM, but to an extended object representing a *family* of POVMs, thus establishing a direct connection with the theory of programmable measurement devices [14, 15, 16]. Third, our sharpness monotones provide a *complete comparison* (in the sense of Blackwell [17, 18]), in that one POVM is sharper than another with respect to all such monotones if and only if the former can be transformed into the latter by means of an appropriate free operation.

The paper is organized as follows. In Section 2 we introduce notations and basic definitions, and review the pre- and postprocessing preorders of POVMs [19]. In Section 3 we define the set of free operations and show that they can be regarded as preprocessings on objects that extend POVMs to programmable devices. In Section 4 we review the theory of EPR–Ozawa measurement correlations [12, 13] and use it to define a class of sharpness measures, which are by construction non-increasing under free operations. In Section 5 we prove a Blackwell–like theorem for sharpness, stating the equivalence between the sharpness preorder, arising from comparing all sharpness measures for a pair of POVMs, and the existence of a sharpness-non-increasing transformation from one POVM into the other one. Finally, in Section 6 we conclude the paper with a summary of our resource theory of sharpness.

2 Quantum measurements and preorders

Let us consider a quantum system A associated with a finite d_A-dimensional Hilbert space $\mathscr{H}_A$. States of A are in one-to-one correspondence with *density matrices* on $\mathscr{H}_A$, i.e., matrices $\rho_A \geqslant 0$ such that $\mathrm{Tr}[\rho_A] = 1$. A quantum state ρ_A contains all the information needed to predict the statistics of any observation done on it, as modeled by a positive operator-valued measure (*POVM*), namely, a family $\mathsf{P} = \{P_A^x\}_{x\in\mathcal{X}}$ of positive semi-definite operators

$P_A^x \geqslant 0$ labeled by the outcome set $\mathcal{X}$ (also assumed to be finite), such that the *completeness relation* $\sum_{x\in\mathcal{X}} P_A^x = \mathbb{1}_A$, where $\mathbb{1}$ denotes the identity matrix, is satisfied. For notational convenience, we will sometimes simply take the outcome set $\mathcal{X}$ to be a subset of the natural numbers, i.e., $\mathcal{X} = \{1, 2, \ldots, N\}$.

The interpretation of POVMs in terms of quantum measurements is based on the Born rule, which postulates that to any observation with outcomes in set $\mathcal{X}$, there corresponds a POVM so that, if the state of the system is given by ρ_A, the expected probability of occurrence of each outcome $x \in \mathcal{X}$ is computed as $\mathrm{Tr}[P_A^x\ \rho_A]$. Notice that, in general, a POVM may contain some null elements, corresponding to the situation in which some outcomes in $\mathcal{X}$ never occur. Further, a POVM is said to be

- *rank-one*, whenever all its elements P_A^x are non-zero and proportional to rank-one projectors;
- *projective*, whenever all its elements P_A^x are non-zero orthogonal projectors, i.e., $P_A^x P_A^{x'} = \delta_{x,x'} P_A^x$;
- *trivial*, whenever each element P_A^x is proportional to the identity matrix, or is the zero matrix.

Any POVM on $\mathscr{H}_A$ with outcome set $\mathcal{X}$ can also be understood as a linear map from the set of density matrices on $\mathscr{H}_A$ to the set of normalized probability distributions on $\mathcal{X}$. More generally, in operational quantum theory a crucial role is played by completely positive trace-preserving (CPTP) linear maps, also known as *quantum channels*, that is, linear maps transforming density matrices on an input space $\mathscr{H}_A$ to density matrices on an output space $\mathscr{H}_B$, in such a way that parallel compositions are well-defined[1]. We will denote any such a quantum channel as $\mathcal{E} : A \to B$ for short. The Born rule naturally associates to any quantum channel $\mathcal{E} : A \to B$ a *trace-dual channel* $\mathcal{E}^\dagger : B \to A$, which maps POVMs on $\mathscr{H}_B$ to POVMs on $\mathscr{H}_A$ and is defined by the equality $\mathrm{Tr}\big[\mathcal{E}^\dagger(Y)\ X\big] := \mathrm{Tr}[Y\ \mathcal{E}(X)]$, for all linear operators Y on $\mathscr{H}_B$ and X on $\mathscr{H}_A$. It is easy to verify that a linear map $\mathcal{E} : A \to B$ is completely positive and trace-preserving if and only if its trace-dual $\mathcal{E}^\dagger : B \to A$ is completely positive and unit-preserving, i.e., $\mathcal{E}^\dagger(\mathbb{1}_B) = \mathbb{1}_A$.

Definition 1 (sharp POVMs)**.** *A POVM* $\mathsf{P} = \{P_A^x\}_{x\in\mathcal{X}}$ *is called* sharp*, whenever all its elements contain the real unit among their eigenvalues, i.e., there exist normalized vectors* $|\psi^x\rangle_A$ *such that* $P_A^x|\psi^x\rangle_A = |\psi^x\rangle_A$ *for all* $x \in \mathcal{X}$*.*

[1] In particular, the notion of *complete positivity* is required so that a quantum channel $\mathcal{E}$ remains a quantum channel even if it is extended with the identity map as $\mathcal{E} \otimes \mathsf{id}$, for any ancillary system.

Due to the completeness relation, we also have that $P_A^{x'}|\psi^x\rangle_A = 0$ for all $x' \neq x$, meaning that the elements of a sharp POVM can be perfectly "discriminated", thus leading to a maximum informational power [20, 21]. The completeness relation also implies that the cardinality of the outcome set of a sharp POVM cannot exceed the dimension of the underlying Hilbert space, i.e., $|\mathcal{X}| \leqslant d_A$. POVMs that are not sharp are usually called *unsharp* or *fuzzy*.

Notice that our definition of sharp POVMs differs from the usual one, see e.g. [7, 9, 10], where sharp POVMs are instead defined as the projective ones, i.e., those whose elements are all non-zero projectors, i.e., $(P_A^x)^2 = P_A^x \neq 0$ for all $x \in \mathcal{X}$. More precisely, while all projective POVMs are sharp according the our definition, the vice versa does not hold. However, sharp POVMs are exactly those that possess the principal operational property of projective POVMs, i.e., their being measurable in a repeatable way. In fact, if repeatability is regarded as *the* reason making projective POVMs "special", then, it is sharpness (as defined here) and not projectivity the right concept to consider: indeed, as discussed for example in Section II.3.5 of [22], a POVM is sharp (as defined here) if and only if it admits a repeatable measurement. In particular, there exist sharp non-projective POVMs that admit a repeatable measurement[2]. Nonetheless, in what follows we will also see how conventional observables, i.e., rank-one projective POVMs, can be singled out as the *minimal sharp POVMs*, for any given outcome set.

2.1 Preorders of POVMs

Two preorders that are relevant for the study of the mathematical properties of POVMs, including their sharpness, are the *quantum preprocessing preorder* and the *classical postprocessing preorder* [19], which are defined as follows.

Given two POVMs $\mathbf{P} = \{P_A^x\}_{x\in\mathcal{X}}$ and $\mathbf{Q} = \{Q_B^x\}_{x\in\mathcal{X}}$, possibly defined on different Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ but with the same outcome set $\mathcal{X}$, we say that $\mathbf{P}$ is *preprocessing cleaner* than $\mathbf{Q}$, whenever there exists a quantum channel $\mathcal{E} : B \to A$ such that $\mathcal{E}^\dagger(P_A^x) = Q_B^x$ for all $x \in \mathcal{X}$.

Further, given two POVMs $\mathbf{P} = \{P_A^x\}_{x\in\mathcal{X}}$ and $\mathbf{Q} = \{Q_A^y\}_{y\in\mathcal{Y}}$, possibly with different outcome sets but defined on the same Hilbert space $\mathscr{H}_A$, we say that $\mathbf{P}$ is *postprocessing cleaner* than $\mathbf{Q}$, whenever there exists a conditional

[2] With the difference that the corresponding repeatable measurement may not be of the von Neumann–Lüders (or "square-root") type [1, 23], but rather of the Gordon–Louisell (or "measure-and-prepare") type [24]. In order to discuss further the notion of repeatability one should employ the concept of *quantum instruments*, but this point is beyond the scope of the present paper. We refer the interested reader to [25, 22, 26] for a careful presentation of the problem and some fundamental results.

probability distribution $\mu(y|x)$ such that $Q_A^y = \sum_x \mu(y|x) P_A^x$ for all $y \in \mathcal{Y}$.

Remark 1. Notice that a necessary condition for P to be preprocessing cleaner than Q is that if, for some $x \in \mathcal{X}$, $P_A^x = 0$, then also $Q_B^x = 0$. That is, outcomes that never occur for P cannot occur for Q either. This is a consequence of the fact that $\mathcal{E}^\dagger$ is linear element-wise, that is, on each POVM element. Instead, the classical postprocessing preorder is more flexible: for example, it may swap an outcome corresponding to a null POVM element with an outcome corresponding to a non-zero operator.

The connection between sharpness and POVMs preorders arises from the following result proved in Ref. [19].

Theorem 1. *A POVM $\mathsf{P} = \{P_A^x\}_{x\in\mathcal{X}}$, defined on a Hilbert space $\mathscr{H}_A$ and with outcome set $\mathcal{X}$, is sharp if and only if P is preprocessing cleaner than any other POVM with the same outcome set $\mathcal{X}$.*

Remark 2. As we noticed already, a necessary condition to be sharp is that $d_A \geqslant |\mathcal{X}|$. Hence, among all sharp POVMs, those that are "minimal" are sharp POVMs defined on a Hilbert space $\mathscr{H}_A$ with $d_A = |\mathcal{X}|$, and these can only be rank-one projective POVMs. In this way, the quantum preprocessing preorder is able to single out conventional non-degenerate observables as its *minimal greatest elements*.

Remark 3. While Theorem 1 characterizes the greatest elements of the quantum preprocessing preorder as sharp POVMs, it is also easy to identify its smallest (or "maximally fuzzy") elements as trivial POVMs[3]. This is due to the fact that the map $\mathcal{E}^\dagger$ is linear and unit-preserving, so that any trivial POVM on A, such as $\{p(x)\mathbb{1}_A\}_{x\in\mathcal{X}}$ for some probability distribution $p(x)$, cannot be transformed into anything that is not trivial. In fact, using quantum preprocessing channels, any trivial POVM can only be mapped into itself (apart from changing the system). This is an indication that a resource theory of sharpness must include more general operations than just quantum preprocessing channels. We will return to this point in the next section.

Proof of Theorem 1. We briefly recount here the proof of the above theorem for the sake of completeness. We begin by showing that any sharp POVM is a greatest element for the quantum preprocessing preorder. If a POVM $\mathsf{P} = \{P_A^x\}_{x\in\mathcal{X}}$ is sharp, then there exist normalized vectors $|\psi^x\rangle_A$ such that

[3] When the outcome set $\mathcal{X}$ is a singleton, there exists only one POVM, which is at once greatest and smallest, sharp and trivial.

$P_A^x|\psi^{x'}\rangle_A = \delta_{x,x'}|\psi^{x'}\rangle_A$. This condition in particular implies that the normalized vectors $|\psi^x\rangle_A$ are also mutually orthogonal. Consider then the linear operator from $\mathscr{H}_B$ to $\mathscr{H}_A \otimes \mathscr{H}_B$

$$V := \sum_{x\in\mathcal{X}} |\psi^x\rangle_A \otimes \sqrt{Q_B^x}\ .$$

Notice that it may be that, for some x, $Q_B^x = 0$. (Instead, the POVM $\mathbf{P}$ is assumed to be sharp.) It is easy to check that V is an isometry, since $V^\dagger V = \sum_x Q_B^x = \mathbb{1}_B$. Moreover, by direct inspection,

$$V^\dagger(P_A^x \otimes \mathbb{1}_B)V = Q_B^x\ ,$$

for all $x \in \mathcal{X}$. Since the linear map $V^\dagger(\bullet_A \otimes \mathbb{1}_B)V$ is by construction completely positive and identity-preserving, the above equation shows that $\mathbf{P}$ is preprocessing cleaner than $\mathbf{Q}$, for any $\mathbf{Q}$, as claimed.

Conversely, let us suppose that $\mathbf{P}$ is preprocessing cleaner than $\mathbf{Q}$, for any other POVM $\mathbf{Q}$. This is equivalent to say that, however we choose the POVM elements Q_B^x, there exists a completely positive unit-preserving linear map $\mathcal{E}^\dagger : A \to B$ such that $\mathcal{E}^\dagger(P_A^x) = Q_B^x$ for all $x \in \mathcal{X}$. Let then Q_B^x constitute a sharp POVM, that is, all Q_B^x's have the real number one as an eigenvalue. But since a completely positive unit-preserving linear map is spectrum-width decreasing (see, e.g., Ref. [19]), the real unit must already be an eigenvalue also of all P_A^x's, whence their sharpness. ■

Instead, the classical postprocessing preorder is not related with the sharpness of POVMs, but rather to their being rank-one or not [27, 19]. For example, a POVM with repeated elements like the following

$$\left\{\frac{1}{2}|\psi^1\rangle\langle\psi^1|_A,\ \frac{1}{2}|\psi^1\rangle\langle\psi^1|_A,\ \frac{1}{2}|\psi^2\rangle\langle\psi^2|_A,\ \frac{1}{2}|\psi^2\rangle\langle\psi^2|_A, \dots\right\}\ ,$$

is a postprocessing clean, even though it is obviously unsharp. This arguably is the reason why attempts to characterize POVMs sharpness using classical postprocessings can only be partially successful, as noticed in [10].

3 Fuzzifying operations

In the light of Theorem 1 and Remark 3, it is tempting to conclude that *sharpness-non-increasing* or *fuzzifying* operations exactly coincide with quantum preprocessing channels. This however cannot be the case for a resource theory of sharpness, as the following example shows. Let us consider two trivial POVMs, such as $\{0, \mathbb{1}\}$ and $\{\mathbb{1}, 0\}$. Since any quantum preprocessing channel is linear and unit-preserving, as noticed in Remark 3, it is impossible to transform $\{0, \mathbb{1}\}$ into $\{\mathbb{1}, 0\}$ or vice versa, since both 0 and $\mathbb{1}$ are fixed points for any linear unit-preserving map. In fact, any trivial POVM on system A can only be transformed into the corresponding trivial POVM on B. This simple observation leads us to conclude that, if free operations were given only by quantum preprocessing channels, the resulting resource theory would have many inequivalent smallest, i.e., resource-free, elements. Instead, one would like a resource theory of sharpness to have *all* trivial POVMs *equivalent* to each other, following the prescription that resource-free objects in a resource theory should all be freely available under free operations [11].

We thus introduce the following definition:

Definition 2 (sharpness preorder)**.** *Given two POVMs $\mathsf{P} = \{P_A^x\}_{x\in\mathcal{X}}$ and $\mathsf{Q} = \{Q_B^x\}_{x\in\mathcal{X}}$, possibly defined on different Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ but with the same outcome set $\mathcal{X}$, we say that P is* sharper *than Q, and write*

$$\mathsf{P} \succeq_{\mathcal{X}}^{\text{sharp}} \mathsf{Q} , \tag{1}$$

whenever there exists a quantum channel $\mathcal{E} : B \to A$, a trivial POVM $\{p(x)\mathbb{1}_B\}_{x\in\mathcal{X}}$ on B, and a number $\mu \in [0, 1]$, such that

$$Q_B^x = \mu\mathcal{E}^\dagger(P_A^x) + (1 - \mu)p(x)\mathbb{1}_B ,$$

for all $x \in \mathcal{X}$.

In other words, denoting by $\mathsf{T}^{(i)} = \{T_B^{x|i}\}_{x\in\mathcal{X}}$ the extremal trivial POVM on B such that $T_B^{x|i} = \delta_{x,i}\mathbb{1}_B$ for all $x, i \in \mathcal{X}$, then P is sharper than Q if and only if Q belongs to the convex hull of $\{\mathcal{E}^\dagger(\mathsf{P})\} \cup \{\mathsf{T}^{(i)}\}_{i\in\mathcal{X}}$. The above definition suggests the following

Definition 3 (fuzzifying operations)**.** *Given a POVM $\mathsf{P} = \{P_A^x\}_{x\in\mathcal{X}}$, a fuzzifying operation on P is any transformation of the form*

$$\forall x \in \mathcal{X} , \qquad P_A^x \mapsto \mu\mathcal{E}^\dagger(P_A^x) + (1 - \mu)p(x)\mathbb{1}_B , \tag{2}$$

for some arbitrary but fixed probability $\mu \in [0, 1]$, probability distribution $p(x)$, and quantum preprocessing channel $\mathcal{E}^\dagger : A \to B$.

Then, Definition 2 can be reformulated as follows: $\mathsf{P} \succeq_{\mathcal{X}}^{\text{sharp}} \mathsf{Q}$ if and only if there exists a fuzzifying operation transforming P into Q. Hence, as it was the case for the quantum preprocessing preorder, we can see that the greatest elements of $\succeq_{\mathcal{X}}^{\text{sharp}}$ are all equivalent and coincide with sharp POVMs. Now, however, also all the smallest elements, i.e., trivial POVMs, turn out to be equivalent to each other. This solves the problem that we raised in Remark 3.

But at this point another problem arises: fuzzifying operations, seen as maps acting on the POVM elements P_A^x, are in general *not* linear, since they could transform zero operators into non-zero operators. But neither they are combinations of quantum preprocessing and classical postprocessing[4] of the POVM P. Thus, the question is how fuzzifying operations can be understood *operationally*. The answer is given by the following construction.

Given a finite outcome set $\mathcal{X} = \{1, 2, \dots, N\}$, we take as the *objects* of the theory not just POVMs with outcome set $\mathcal{X}$, but rather *families* comprising $N+1$ POVMs: the first POVM, which is the given POVM P whose sharpness is being evaluated, together with the N extremal trivial POVMs $\mathsf{T}^{(i)}$, with $i = 1, \dots, N$, introduced above. More explicitly, given a POVM $\mathsf{P} = \{P_A^x\}_{x\in\mathcal{X}}$, the corresponding object in the resource theory is given by the family

$$\begin{aligned}\overline{\mathsf{P}} &\equiv \{\mathsf{P}_0, \mathsf{P}_1, \mathsf{P}_2, \dots, \mathsf{P}_N\} \\ &:= \{\mathsf{P}, \mathsf{T}^{(1)}, \mathsf{T}^{(2)}, \dots \mathsf{T}^{(N)}\} \\ &= \left\{ \begin{pmatrix} P_A^1 \\ P_A^2 \\ \vdots \\ P_A^N \end{pmatrix}, \begin{pmatrix} \mathbb{1}_A \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbb{1}_A \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \mathbb{1}_A \end{pmatrix} \right\} .\end{aligned} \tag{3}$$

Notice that there is a one-to-one correspondence between POVMs P and extended families $\overline{\mathsf{P}}$. Hence, in what follows, when writing $\overline{\mathsf{P}}$ we will understand it as the family of $N+1$ POVMs that have the POVM P as its first element, and the N extremal trivial POVMs, in the same order as in Eq. (3).

Since $\overline{\mathsf{P}}$ is a family of POVMs, following [14] we regard it as a *programmable* POVM, where the program is an element i of the set $\mathcal{I} := \{0\} \cup \mathcal{X} = \{0, 1, 2, \dots, N\}$. As schematically depicted in Fig. 1, a programmable POVM is a device with two inputs and one output: one quantum input A, i.e., the quantum system being measured, one classical input $i \in \mathcal{I}$, i.e. the program deciding which POVM to measure, and one classical output, i.e., the outcome $x \in \mathcal{X}$.

We now want to show that fuzzifying operations on P can be seen as preprocessing channels applied on the extended object $\overline{\mathsf{P}}$. We do this by defining preprocessing channels on $\overline{\mathsf{P}} : A \times \mathcal{I} \to \mathcal{X}$ in such a way that the resulting device $\overline{\mathsf{Q}} : B \times \mathcal{I} \to \mathcal{X}$ is such that

1. $Q_B^{x|0} = \mathcal{E}^\dagger \left(\sum_{i=0}^N \mu(i) P_A^{x|i} \right)$, for all $x \in \mathcal{X}$;

[4] In fact, as noticed in Ref. [10], classical postprocessing can increase sharpness, and thus cannot be part of sharpness-non-increasing operations.

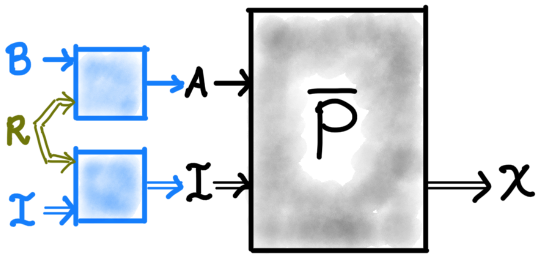

Figure 1: Given a POVM $\mathbf{P}$ with outcome set $\mathcal{X}$, we uniquely associate to it a programmable POVM $\overline{\mathbf{P}}$ (in black). Correspondingly, any fuzzifying operation on $\mathbf{P}$ is uniquely associated with an extended preprocessing on $\overline{\mathbf{P}}$ (in blue). The quantum preprocessing channel $B \to A$ is an arbitrary completely positive trace-preserving linear map, while the classical preprocessing on the program alphabet $\mathcal{I} := \{0\} \cup \mathcal{X}$ is restricted to act as the identity channel on all program values different from zero. Free shared randomness between the two local preprocessing channels is allowed (represented by the green classical random variable R).

2. $Q_B^{x|i} = \delta_{x,i}\mathbb{1}_B$, for all $x \in \mathcal{X}$ and all $i \in \{1, \ldots, N\}$.

Notice that the above condition $Q_B^{x|0} = \mathcal{E}^\dagger\left(\sum_{i=0}^N \mu(i) P_A^{x|i}\right)$ can be rewritten as Eq. (2), with the replacements $\mu \leftrightarrow \mu(0)$, so that $(1-\mu) \leftrightarrow \sum_{x=1}^N \mu(x)$, and $p(x) \leftrightarrow \frac{\mu(x)}{\sum_{x=1}^N \mu(x)}$.

The above structure can be obtained if we take as free operations (again, refer to Fig. 1 for a diagram):

1. all quantum preprocessing channels $\mathcal{E}$ mapping system B into system A;
2. all classical preprocessing channels (i.e., conditional probabilities) from $\mathcal{I}$ to $\mathcal{I}$, acting identically on $\mathcal{I} \setminus \{0\}$;
3. convex combinations of any parallel compositions of the above.

We call the above class of preprocessing channels *extended fuzzifying preprocessings*. Notice that these are all completely positive trace-preserving linear maps by construction.

Hence, we have seen that fuzzifying operations given in Def. 3, even though they are not linear in $\mathbf{P}$, they can nonetheless be regarded as linear maps, more precisely, as extended fuzzifying preprocessings, acting on the programmable device $\overline{\mathbf{P}}$. We summarize this discussion as follows:

Theorem 2. *Given two POVMs* $\mathsf{P} = \{P_A^x\}_{x\in\mathcal{X}}$ *and* $\mathsf{Q} = \{Q_B^x\}_{x\in\mathcal{X}}$*, possibly defined on different Hilbert spaces* $\mathscr{H}_A$ *and* $\mathscr{H}_B$ *but with the same outcome set* $\mathcal{X}$*,* $\mathsf{P} \succeq_{\mathcal{X}}^{\text{sharp}} \mathsf{Q}$ *if and only if there exists an extended fuzzifying preprocessing transforming* $\overline{\mathsf{P}}$ *into* $\overline{\mathsf{Q}}$*.*

4 EPR–Ozawa measurement correlations

In order to clarify in a mathematically rigorous way the meaning of the statement, crucial for the EPR argument [28], that "two observables have the same value", Ozawa introduced the concept of quantum perfect correlations [12, 13].

Definition 4. *Given a state* ρ_A *on* $\mathscr{H}_A$ *and two POVMs on* A *with the same outcome set* $\mathcal{X}$*,* $\mathsf{P} = \{P_A^x\}_{x\in\mathcal{X}}$ *and* $\mathsf{R} = \{Z_A^x\}_{x\in\mathcal{X}}$*, we say that* P *and* R *are* perfectly correlated *in state* ρ *if*

1. *they are* jointly distributed *in* ρ*, that is,* $\mathrm{Tr}\left[P_A^x Z_A^{x'}\ \rho_A\right] \geqslant 0$ *for all* $x, x' \in \mathcal{X}$*, and*
2. $\sum_{x\in\mathcal{X}} \mathrm{Tr}[P_A^x Z_A^x\ \rho_A] = 1$.

More generally, if two POVMs P *and* R *are jointly distributed in state* ρ*, their* degree of correlation *is defined as*

$$\kappa_\rho(\mathsf{P} : \mathsf{R}) := \sum_{x\in\mathcal{X}} \mathrm{Tr}[P_A^x Z_A^x\ \rho_A]\ .$$

Remark 4. The sharpness measure $\mathcal{P}^L(\rho; \mathsf{P}) := \sum_x \mathrm{Tr}[\rho_A\ (P_A^x)^2]$ introduced in Eq. (6) of Ref. [10], is a special case of our degree of correlation: more precisely, it coincides with the degree of *autocorrelation* $\kappa_\rho(\mathsf{P} : \mathsf{P})$.

In what follows we will consider in particular the case in which the state ρ_A is maximally mixed. In such a case, any two POVMs are always jointly distributed, so that the degree of correlation can be discussed without further assumptions. In that case, we will use the short-hand notation

$$\kappa_u(\mathsf{P} : \mathsf{R}) := \frac{1}{d_A} \sum_{x\in\mathcal{X}} \mathrm{Tr}[P_A^x\ Z_A^x]\ , \tag{4}$$

where the subscript u stands for the "uniform", i.e., maximally mixed state $\frac{1}{d_A}\mathbb{1}_A$. Notice that, in this case, the degree of correlation can also be written as follows:

$$\kappa_u(\mathsf{P} : \mathsf{R}) = \sum_{x\in\mathcal{X}} \mathrm{Tr}\left[({}^t P_{A'}^x \otimes Z_A^x)\ |\Phi^+\rangle\langle\Phi^+|_{A'A}\right]\ ,$$

where $|\Phi^+\rangle_{A'A} := \frac{1}{\sqrt{d_A}} \sum_{i=1}^{d_A} |i\rangle_{A'} \otimes |i\rangle_A$ is the maximally entangled state between A and an auxiliary system $A' \cong A$, and the left-hand superscript ${}^t\bullet$ denotes the transposition done with the respect to the basis $\{|i\rangle\}_i$ used in the definition of $|\Phi^+\rangle_{A'A}$. This shows that the degree of correlation $\kappa_u(\mathsf{P} : \mathsf{R})$ is, in principle, experimentally observable for any pair of POVMs.

4.1 The "tuning" preorder

When computing the degree of uniform correlation in Eq. (4), let us imagine that the POVM R plays the role of a "reference measurement", with respect to which the outcome set is fixed. Let us hence consider a reference POVM $\mathsf{R} = \{Z_R^x\}_{x\in\mathcal{X}}$, with outcome set $\mathcal{X}$ and defined on some reference system R with Hilbert space $\mathscr{H}_R$.

Fixed a reference POVM $\mathsf{R} = \{Z_R^x\}_{x\in\mathcal{X}}$, we can now measure how another POVM, say, $\mathsf{P} = \{P_A^x\}_{x\in\mathcal{X}}$, with the same outcome set of the reference but otherwise arbitrary, can be "tuned" with the reference R. Here, we focus on the following quantity

$$\begin{aligned}\kappa_u^*(\mathsf{R}|\mathsf{P}) : &= \max_{\mathcal{L}} \kappa_u(\mathcal{L}(\mathsf{P}) : \mathsf{R}) \\ &= \max_{\mathcal{L}} \frac{1}{d_R} \sum_{x\in\mathcal{X}} \mathrm{Tr}[\mathcal{L}(P_A^x)\ Z_R^x]\ ,\end{aligned} \tag{5}$$

where the optimization is done over all fuzzifying operations $\mathcal{L}$, as given in Eq. (2). In other words, the quantity $\kappa_u^*(\mathsf{R}|\mathsf{P})$ measures the degree of uniform correlations that can be established between a given POVM P and the reference R by means of a sharpness-non-increasing operation applied on P. We will refer to the quantity $\kappa_u^*(\mathsf{R}|\mathsf{P})$ as the *tuning degree* of P with respect to R. Notice that while the degree of correlation (4) is symmetric in the POVMs, the tuning degree (5) is not, since the optimization is done only on one of the two POVMs. The notation $\kappa_u^*(\mathsf{R}|\mathsf{P})$ reflects this. Notice also that other choices for the tuning process may be done: this freedom is similar to what happens, for example, in the resource theory of entanglement, for which there exist different, though all operationally meaningful, notions of entanglement manipulation, such as LOCC [29] or LOSR [30]. However, in the context of the present paper it is natural to define the optimization with respect to the same class of transformations that is used to define the sharpness preorder $\succeq_{\mathcal{X}}^{\text{sharp}}$ in Definition 2.

From the definition (5), it is clear that a sharp POVM P allows for *ideal* tuning. Since, as Theorem 1 states, sharp POVMs are exactly those that can be transformed into *any other* POVM by means of a suitable fuzzifying operation (in fact, a linear quantum preprocessing suffices), the quantity $\kappa_u^*(\mathbf{R}|\mathbf{P})$, if $\mathbf{P}$ is sharp, can be pushed up to its maximum value, namely,

$$\kappa_u^*(\mathbf{R}) := \max_{\{\tilde{Z}_R^x\}_x\text{: POVM}} \frac{1}{d_R} \sum_{x\in\mathcal{X}} \mathrm{Tr}\left[\tilde{Z}_R^x \; Z_R^x\right] \;, \tag{6}$$

which now only depends on the reference $\mathbf{R}$. But even if the POVM $\mathbf{P}$ is not sharp, the tuning degree $\kappa_u^*(\mathbf{R}|\mathbf{P})$, for any given reference $\mathbf{R}$, can still be considered as a measure of the sharpness of $\mathbf{P}$ (see also Remark 4 above).

Remark 5. Notice that the quantity appearing in Eq. (6) is equal to the maximum probability of correctly discriminating among the states of the ensemble $\{p(x), \rho(x)\}_{x\in\mathcal{X}}$, where $p(x) := \frac{1}{d_R}\mathrm{Tr}[Z_R^x]$ and $\rho_R^x := \frac{1}{\mathrm{Tr}[Z_R^x]} Z_R^x$. It can thus be effectively computed using a simple semi-definite program, see for example Section 3.1.2 of Ref. [31]. Moreover, by using the theory of *pretty good measurements* [32, 33] , it is also possible to show that

$$\kappa_u^*(\mathbf{R}) \geqslant \kappa_u(\mathbf{R} : \mathbf{R}) \geqslant 2\kappa_u^*(\mathbf{R}) - 1 \;,$$

that is, the degree of autocorrelation considered in Remark 4 constitutes a "pretty good" estimate of $\kappa_u^*(\mathbf{R})$, whenever $\kappa_u^*(\mathbf{R})$ is large enough.

We now use the operational task of tuning to compare two POVMs with the same outcome set as follows.

Definition 5 (tuning preorder). *Given a reference POVM $\mathbf{R} = \{Z_R^x\}_{x\in\mathcal{X}}$ and two POVMs $\mathbf{P} = \{P_A^x\}_{x\in\mathcal{X}}$ and $\mathbf{Q} = \{Q_B^x\}_{x\in\mathcal{X}}$, possibly defined on different Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ but with the same outcome set as the reference $\mathbf{R}$, we say that $\mathbf{P}$ is* more tunable *than $\mathbf{Q}$ with respect to $\mathbf{R}$, and write*

$$\mathbf{P} \succeq_{\mathbf{R}}^{\mathrm{t}} \mathbf{Q} \;, \tag{7}$$

whenever $\kappa_u^*(\mathbf{R}|\mathbf{P}) \geqslant \kappa_u^*(\mathbf{R}|\mathbf{Q})$.

Further, given two POVMs $\mathbf{P} = \{P_A^x\}_{x\in\mathcal{X}}$ and $\mathbf{Q} = \{Q_B^x\}_{x\in\mathcal{X}}$, possibly defined on different Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ but with the same outcome set $\mathcal{X}$, we say that $\mathbf{P}$ is always more tunable *than $\mathbf{Q}$, and write*

$$\mathbf{P} \succeq_{\mathcal{X}}^{\mathrm{t}} \mathbf{Q} \;, \tag{8}$$

whenever $\mathbf{P} \succeq_{\mathbf{R}} \mathbf{Q}$ for all reference POVMs $\mathbf{R}$ with outcome set $\mathcal{X}$.

5 Equivalence of comparisons

In this section we develop the theory of statistical comparison for the sharpness preorder that we introduced above. Statistical comparison is a concept introduced by Blackwell [17] with the aim of extending the ideas of Lorenz curves and majorization [34, 35] to more general scenarios. It establishes an equivalence between two kinds of preorders: a "sufficiency" preorder, analogous to the majorization preorder, which is given by the existence of a suitable transformation (e.g., a doubly stochastic matrix, in the case of majorization) between two objects; and a "game-theoretic" preorder, which instead concerns the comparison of the expected performance with respect to a certain class of statistical tests (e.g., hypothesis testing, in the case of Lorenz curves). Such an equivalence between, on the one hand, the existence of a transformation and, on the other hand, the comparison of operational utilities (i.e., the "monotones" of resource theories), summarizes the core concept that lies at the basis of all resource theories [11]. In this spirit, various generalizations of Blackwell's theory of statistical comparison [18, 36, 37, 38, 39, 40, 41] have been successfully applied in several specific resource-theoretic scenarios, including entanglement and nonlocality theory [30, 42, 43, 44], quantum communication theory [45, 46], open quantum systems dynamics [47, 48], quantum coherence [49], quantum thermodynamics [50, 51], and quantum measurement theory [36, 52, 14, 15, 16].

The Blackwell–like theorem that we prove in this work is the following.

Theorem 3. *Given two POVMs $\mathsf{P} = \{P_A^x\}_{x\in\mathcal{X}}$ and $\mathsf{Q} = \{Q_B^x\}_{x\in\mathcal{X}}$, possibly defined on different Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ but with the same outcome set $\mathcal{X}$, P can be transformed into Q by means of a fuzzifying operation, that is,*

$$\mathsf{P} \succeq_{\mathcal{X}}^{\text{sharp}} \mathsf{Q} \; ,$$

if and only if P is always more tunable than Q, that is,

$$\mathsf{P} \succeq_{\mathcal{X}}^{\text{t}} \mathsf{Q} \; . \tag{9}$$

Moreover, the comparison (9) can be restricted without loss of generality to reference POVMs defined on the same Hilbert space as Q, i.e., $\mathscr{H}_B$.

Hence, the tuning degrees $\kappa_u^*(\mathsf{R}|\mathsf{P})$, for varying reference POVM R, provide a complete set of monotones for the resource theory of sharpness.

Remark 6. In fact, the proof shows that another condition, at first sight weaker, is in fact equivalent to (9), i.e.,

$$\kappa_u^*(\mathsf{R}|\mathsf{P}) \geqslant \kappa_u(\mathsf{Q} : \mathsf{R}) \; , \tag{10}$$

for all reference POVMs $\mathbf{R} = \{Z_B^x\}_{x\in\mathcal{X}}$. Notice that the right-hand side in the above equation is *not* optimized.

Proof of Theorem 3. Our aim is to show that the condition about the existence of a fuzzifying operation transforming $\mathbf{P}$ into $\mathbf{Q}$, can be equivalently written as Eq. (9).

From Theorem 2, we know that $\mathbf{P} \succeq_{\mathcal{X}}^{\text{sharp}} \mathbf{Q}$, if and only if there exists a fuzzifying preprocessing transforming the extended programmable device corresponding to $\mathbf{P}$, i.e., $\overline{\mathbf{P}}$, into the extended programmable device corresponding to $\mathbf{Q}$, i.e., $\overline{\mathbf{Q}}$. For notational convenience, let us denote the fuzzifying preprocessing as $\mathcal{L}$ and the elements of the resulting programmable device as $\mathcal{L}(\overline{\mathbf{P}})_B^{x|i}$.

Let us now consider an arbitrary but fixed complete set of density matrices $\{\gamma_B^b\}_{b\in\mathcal{B}}$, in the sense that the linear span of $\{\gamma_B^b\}_{b\in\mathcal{B}}$ coincides with the set of all linear operators on $\mathscr{H}_B$. Then, $\mathbf{P} \succeq_{\mathcal{X}}^{\text{sharp}} \mathbf{Q}$ if and only if

$$\operatorname{Tr}\left[\mathcal{L}(\overline{\mathbf{P}})_B^{x|i}\ \gamma_B^b\right] = \operatorname{Tr}\left[Q_B^{x|i}\ \gamma_B^b\right] , \qquad \forall x, \forall i, \forall b . \tag{11}$$

Looking at the two conditional distributions above as vectors in $\mathbb{R}^{|\mathcal{X}|\times|\mathcal{I}|\times|\mathcal{B}|}$, that is, $\boldsymbol{p}_{\mathcal{L}}$ and $\boldsymbol{q}$, respectively, let us consider the subset of $\mathbb{R}^{|\mathcal{X}|\times|\mathcal{I}|\times|\mathcal{B}|}$ defined as

$$\mathcal{C}(\overline{\mathbf{P}}) := \left\{\boldsymbol{p}_{\mathcal{L}} : p_{\mathcal{L}}(x|i,b) = \operatorname{Tr}\left[\mathcal{L}(\overline{\mathbf{P}})_B^{x|i}\ \gamma_B^b\right]\right\} ,$$

where $\mathcal{L}$ can range over all fuzzifying preprocessings. Then, Eq. (11) can be equivalently rewritten as

$$\boldsymbol{q} \in \mathcal{C}(\overline{\mathbf{P}}) . \tag{12}$$

The crucial observation now is that, since the definition of fuzzifying preprocessings involves free shared randomness, they form a convex set. For this reason, also $\mathcal{C}(\overline{\mathbf{P}})$ is a convex subset of $\mathbb{R}^{|\mathcal{X}|\times|\mathcal{I}|\times|\mathcal{B}|}$. Hence, as a consequence of the separation theorem for convex sets, we can rewrite condition (12) in terms of linear functionals as follows

$$\boldsymbol{\lambda}\cdot\boldsymbol{q} \leqslant \max_{\boldsymbol{p}_{\mathcal{L}}\in\mathcal{C}(\overline{\mathbf{P}})} \boldsymbol{\lambda}\cdot\boldsymbol{p}_{\mathcal{L}} , \qquad \forall\boldsymbol{\lambda}\in\mathbb{R}^{|\mathcal{X}|\times|\mathcal{I}|\times|\mathcal{B}|} ,$$

which, once rewritten in a more explicit form, becomes

$$\max_{\mathcal{L}} \sum_{x,i,b} \lambda_{xib} \operatorname{Tr}\left[\mathcal{L}(\overline{\mathbf{P}})_B^{x|i}\ \gamma_B^b\right] \geqslant \sum_{x,i,b} \lambda_{xib} \operatorname{Tr}\left[Q_B^{x|i}\ \gamma_B^b\right] , \qquad \forall\lambda_{xib}\in\mathbb{R} .$$

Introducing the self-adjoint operators $\Gamma_B^{xi} := \sum_b \lambda_{xib}\gamma_B^b$, the above condition becomes

$$\max_{\mathcal{L}} \sum_{x,i} \mathrm{Tr}\left[\mathcal{L}(\overline{\mathbf{P}})_B^{x|i}\ \Gamma_B^{xi}\right] \geqslant \sum_{x,i} \mathrm{Tr}\left[Q_B^{x|i}\ \Gamma_B^{xi}\right] , \qquad \forall \text{ self-adjoint } \{\Gamma_B^{xi}\}_{x,i} .$$

First, we notice that since, by construction, $\mathcal{L}(\overline{\mathbf{P}})_B^{x|i} = Q_B^{x|i} = \delta_{x,i}\mathbb{1}_B$ for all $x, i \in \mathcal{X}$ and any choice of the fuzzifying process $\mathcal{L}$, we can in fact focus only on the case $i = 0$. Therefore, in what follows, we will only consider the conditions

$$\max_{\mathcal{L}} \sum_{x} \mathrm{Tr}\left[\mathcal{L}(\overline{\mathbf{P}})_B^{x|0}\ \Gamma_B^{x}\right] \geqslant \sum_{x} \mathrm{Tr}[Q_B^{x}\ \Gamma_B^{x}] , \qquad \forall \text{ self-adjoint } \{\Gamma_B^{x}\}_{x} . \tag{13}$$

The next step is to notice that, since $\sum_x \mathcal{L}(\overline{\mathbf{P}})_B^{x|0} = \sum_x Q_B^x = \mathbb{1}_B$, it is possible to shift and rescale the operators Γ_B^x in such a way that, without loss of generality, we can restrict condition (13) to families of operators $\{Z_B^x\}$ such that $\sum_x Z_B^x = \mathbb{1}_B$ and $Z_B^x \geqslant 0$ for all $x \in \mathcal{X}$, i.e., POVMs $\mathbf{R} = \{Z_B^x\}_{x\in\mathcal{X}}$ on B. Hence, we have been able to rewrite condition (13), which we recall is equivalent to $\mathbf{P} \succeq_{\mathcal{X}}^{\text{sharp}} \mathbf{Q}$, as follows:

$$\max_{\mathcal{L}} \sum_{x} \mathrm{Tr}\left[\mathcal{L}(\overline{\mathbf{P}})_B^{x|0}\ Z_B^{x}\right] \geqslant \sum_{x} \mathrm{Tr}[Q_B^{x}\ Z_B^{x}] , \qquad \forall \text{ POVMs } \mathbf{R} = \{Z_B^{x}\}_{x} ,$$

namely,

$$\kappa_u^*(\mathbf{R}|\mathbf{P}) \geqslant \kappa_u(\mathbf{Q} : \mathbf{R}) , \qquad \forall \text{ POVMs } \mathbf{R} = \{Z_B^{x}\}_{x} . \tag{14}$$

Finally, since the inequality $\kappa_u^*(\mathbf{R}|\mathbf{Q}) \geqslant \kappa_u(\mathbf{Q} : \mathbf{R})$ is true by definition, we reach the conclusion that if condition (9) holds, then also condition (14) holds, which in turn is equivalent to (11). Hence, we have proved that if $\mathbf{P} \succeq_{\mathcal{X}}^{\text{t}} \mathbf{Q}$ then $\mathbf{P} \succeq_{\mathcal{X}}^{\text{sharp}} \mathbf{Q}$.

The converse is trivial: if $\mathbf{P} \succeq_{\mathcal{X}}^{\text{sharp}} \mathbf{Q}$ then obviously any tuning degree that can be achieved with $\mathbf{Q}$ can also be achieved with $\mathbf{P}$, simply because the latter can be transformed into the former, and the compositions of fuzzifying operations is again a fuzzifying operation. ■

Since sharp POVMs and trivial POVMs are all equivalent under sharpness-non-increasing operations, we immediately obtain the following:

Corollary 3.1. *All sharp POVMs achieve exactly the same tuning degree for any reference POVM. The same holds for all trivial POVMs. Hence, for any reference POVM* $\mathbf{R} = \{Z_R^x\}_{x\in\mathcal{X}}$ *and any POVM* $\mathbf{P} = \{P_A^x\}_{x\in\mathcal{X}}$*,*

$$\frac{1}{d_R} \max_x \mathrm{Tr}[Z_R^x] \leqslant \kappa_u^*(\mathbf{R}|\mathbf{P}) \leqslant \kappa_u^*(\mathbf{R}) .$$

In particular, any value $\kappa_u^*(\mathsf{R}|\mathsf{P})$ *strictly larger than the trivial lower bound provides a* measurement device-independent witness *of the non-triviality of* P.

6 Summary of the theory

For the reader's convenience, we summarize the main points of the resource theory of sharpness that we have derived.

- The *objects* of the theory are POVMs. In particular, this means that our resource theory of sharpness does not depend on the specific numerical values associated with each measurement outcome, i.e., the observable's eigenvalues, nor on any particular instrument or measurement process used to realize the POVM.

- The *free operations* are given by the class of fuzzifying operations, which is by construction convex and closed under sequential composition (see Definition 3). Though fuzzifying operations are neither quantum preprocessings nor classical postprocessings of the POVM alone, they can be seen as the preprocessing of a programmable measurement device that extends the given POVM in a one-to-one way.

- The *greatest objects* in the resource theory of sharpness are, of course, sharp POVMs, and they are all equivalent, in the sense that any sharp POVM can be freely transformed into any other sharp POVM with the same outcome set. We recall that, in this work, we define sharp POVMs in terms of the operational property of being measurable in a repeatable way (see Definition 1).

- The *smallest objects* are trivial POVMs, i.e., POVMs whose elements are all proportional (including the possibility of zero elements) to the identity operator. As it happens for sharp POVMs, also trivial POVMs are, as one would expect, all equivalent.

- The *sharpness monotones* are given by the tuning degrees $\kappa_u^*(\mathsf{R}|\mathsf{P})$, for varying reference POVM R, defined in Eq. (5).

- A *Blackwell–like theorem* for sharpness holds, i.e., a POVM can be transformed into another POVM by a free operation, if and only if there exists no tuning degree for the latter that is higher than for the former. This automatically implies that all sharp POVMs and all trivial POVMs achieve exactly the same tuning degree for any reference

POVM, as given in Corollary 3.1. By normalizing these two numbers, our sharpness monotones satisfy Busch's requirements for sound sharpness measures [7]. Moreover, sharpness monotones can be used to witness in a *measurement device-independent way* the non-triviality of a POVM, exactly in the same way that semiquantum games can witness non-separability [30, 53].

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